Inequality 10

16 November 2023 18:14

Q) If $a, b, c > 0$ then prove that,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{9}{a+b+c}$$

Ans – $\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)(a+b+c)$

$$= \left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\frac{1}{2}(a+b+b+c+c+a)$$

$$\geq \frac{1}{2}\, 3\sqrt[3]{\frac{1}{(a+b)}\frac{1}{(b+c)}\frac{1}{(c+a)}}\; 3\sqrt[3]{(a+b)(b+c)(c+a)}$$

$$\geq \frac{9}{2}$$

$$\Rightarrow 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{9}{a+b+c}$$

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \frac{2}{\frac{1}{\frac{1}{a}}+\frac{1}{\frac{1}{b}}} \longrightarrow AM \geq HM$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b} \qquad \frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c} \qquad \frac{1}{c}+\frac{1}{a} \geq \frac{4}{c+a}$$

$$\Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 4\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

Q) For real numbers $x, y > 1$ prove that,

$$\frac{x^2}{y-1}+\frac{y^2}{x-1} \geq 8$$

$$\frac{x^2}{\ldots} + \frac{y^2}{\ldots} \geq 2\sqrt{\frac{x^2y^2}{\ldots}} = 2\,\frac{xy}{\ldots}$$

Ans:— $\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 2\sqrt{\frac{x^2y^2}{(x-1)(y-1)}} = 2\frac{xy}{\sqrt{x-1}\sqrt{y-1}}$

$= 2\frac{x}{\sqrt{x-1}}\frac{y}{\sqrt{y-1}}$

$\geq 2\cdot 2\cdot 2$

$= 8$

$\frac{x}{\sqrt{x-1}} \geq 2$

$(x-2)^2 \geq 0$

$x^2-4x+4 \geq 0$

$x^2 \geq 4x-4$

$x^2 \geq 4(x-1)$

$\frac{x^2}{x-1} \geq 4 \Rightarrow \frac{x}{\sqrt{x-1}} \geq 2$

Q> $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Then prove that

$n(n+1)^{\frac{1}{n}} < n+H_n$ for $n \geq 2$

Ans:— $n+H_n = (1+1)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)+\cdots+\left(1+\frac{1}{n}\right)$

$\frac{n+H_n}{n} > \sqrt[n]{\cancel{2}\times\frac{\cancel{3}}{\cancel{2}}\times\frac{\cancel{4}}{\cancel{3}}\times\cdots\times\frac{n+1}{\cancel{n}}} = (n+1)^{\frac{1}{n}}$

$n+H_n > n(n+1)^{1/n}$

Q> Let a, b, c be positive numbers with $a+b+c=1$, then prove that $\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right) \geq 64$

Ans:— $\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right)$

$= \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc}+1$

$\geq 1+3\sqrt[3]{\frac{1}{abc}}+3\sqrt[3]{\frac{1}{(abc)^2}}+\frac{1}{abc}$

$= \left(1+\frac{1}{\sqrt[3]{abc}}\right)^3 \geq 4^3$

$a+b+c \geq 3\sqrt[3]{abc} \Rightarrow \sqrt[3]{abc} \leq \frac{1}{3} \Rightarrow \frac{1}{\sqrt[3]{abc}} \geq 3$

Cauchy-Schwarz Inequality :-

For real numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ we get,

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Equality holds iff $\exists$ some $\lambda \in \mathbb{R}$ such that $x_i = \lambda y_i$ $\forall i \in \{1, 2, \ldots, n\}$

Proof :- HomeWork

Q> For all positive real numbers x, y, z, prove that

$$x^4 + y^4 + z^2 \geq \sqrt{8}\, xyz$$

Ans:- $$x^4 + y^4 + \frac{z^2}{2} + \frac{z^2}{2} \geq 4\sqrt[4]{x^4 y^4 \frac{z^2}{2}\frac{z^2}{2}} = 4\sqrt[4]{\frac{1}{4}}\, xyz$$

$$= 2^2\, 2^{-2/4}\, xyz$$

$$= 2^{3/2}\, xyz$$

$$= \sqrt{8}\, xyz$$

HomeWork :-

Q> Let a, b, c be positive numbers with $a+b+c=1$, prove

Q> Let a, b, c be positive numbers with $a+b+c=1$, prove that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$

Q> Let a, b, c be positive real numbers and $a+b+c=3$, then prove that,

$$\frac{a+1}{b^2+1} + \frac{b+1}{c^2+1} + \frac{c+1}{a^2+1} \geq 3$$

• If $|x^2-7x+12| > x^2-7x+12$ then,

$\Rightarrow x^2-7x+12 < 0$ (as $>$ is strict)

$x \in (3, 4)$